GREEN FUNCTIONS WITH SINGULARITIES ALONG COMPLEX SPACES

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August 30, 2004

Abstract

We study properties of a Green function G_A with singularities along a complex subspace A of a complex manifold X. It is defined as the largest negative plurisubharmonic function u satisfying locally $u \le \log|\psi| + C$, where $\psi = (\psi_1, \ldots, \psi_m)$, $\psi_1, \ldots, \psi_m$ are local generators for the ideal sheaf $\mathcal{I}_A$ of A, and C is a constant depending on the function u and the generators. A motivation for this study is to estimate global bounded functions from the sheaf $\mathcal{I}_A$ and thus proving a "Schwarz Lemma" for $\mathcal{I}_A$.

Subject Classification (2000): Primary 32U35. Secondary 32C15, 32C25, 32H02, 32S45, 32U25, 32U40.

1 Introduction

If φ is a bounded holomorphic function on a complex manifold X, then it is a natural problem to estimate $|\varphi|$ given some information on the location of the zeros of φ and their multiplicities. If $|\varphi| \le 1$ and the only given information is that $\varphi(a) = 0$ for a single point a, then

$$\log|\varphi| \le G_{X,a} = G_X(\cdot, a),$$

where $G_{X,a}$ is the *pluricomplex Green function with logarithmic pole at* a. It is defined as the supremum over the class $\mathcal{F}_{X,a}$ of all negative plurisubharmonic functions u such that $u \le \log|\zeta| + C$ near a, where ζ are local coordinates near a with $\zeta(a) = 0$ and C is a positive constant depending on u and ζ. The function $G_{X,a}$ was introduced and studied by several authors [16], [24], [9], [19], [4], see also [10], [5].

A generalization is to take $A = (|A|, \{m_a\}_{a\in|A|})$, where $|A|$ is a finite subset of X, m_a is a positive real number for every $a \in |A|$, and assume that φ has a zero of multiplicity at least m_a at every point a in $|A|$. Then

$$\log|\varphi| \le G_A,$$

where G_A is the *Green function with several weighted logarithmic poles*. It is defined as the supremum over the class of all negative plurisubharmonic functions u on X satisfying $u \le m_a \log|\zeta_a| + C$ for every a in $|A|$, where ζ_a are local coordinates near a with $\zeta_a(a) = 0$ and C is a positive constant depending on ζ_a and u. The function G_A was first introduced by Zaharyuta [24] and independently by Lelong [15].

The notion of multiplicity of a zero of an analytic function has a natural generalization as a Lelong number of a plurisubharmonic function. If u is plurisubharmonic in some neighbourhood of the origin 0 in $\mathbb{C}^n$, then the *Lelong number* $\nu_u(0)$ of u at 0 can be defined as

$$\nu_u(0) = \lim_{r\to 0} \frac{\sup\{u(x);\ |x|\le r\}}{\log r}$$

and if u is plurisubharmonic on a manifold X then the Lelong number $\nu_u(a)$ of u at $a\in X$ is defined as $\nu_u(a) = \nu_{u\circ\zeta^{-1}}(0)$, where ζ are local coordinates near a with $\zeta(a)=0$. It is clear that this definition is independent of the choice of local coordinates and that $\nu_u(a)$ equals the multiplicity of a as a zero of the holomorphic function φ in the case $u=\log|\varphi|$. Note that the pluricomplex Green function $G_{X,a}$ with logarithmic pole at a can be equivalently defined as the upper envelope of all negative plurisubharmonic functions u on X satisfying $\nu_u(a)\ge 1$ and, similarly, $\nu_u(a)\ge m_a$ for the Green functions with several weighted logarithmic poles.

For any non-negative function α on X, Lárusson and Sigurdsson [11], [12] introduced the Green function $\tilde G_\alpha$ as the supremum over the class $\tilde{\mathcal{F}}_\alpha$ of all negative plurisubharmonic functions with $\nu_u\ge\alpha$. It is clear that if φ is holomorphic on X, $|\varphi|\le 1$, and every zero a of φ has multiplicity at least $\alpha(a)$, then

$$\log|\varphi| \le \tilde G_\alpha.$$

In this context it is necessary to note that we assume that the manifold X is connected, we take the constant function $-\infty$ as plurisubharmonic, and set $\nu_{-\infty}=+\infty$. Hence $-\infty\in\tilde{\mathcal{F}}_\alpha$ for every α. By [11], Prop. 5.1, $\tilde G_\alpha\in\tilde{\mathcal{F}}_\alpha$.

In the special case when X is the unit disc $\mathbb{D}$ in $\mathbb{C}$, we have

$$\tilde G_\alpha(z) = \sum_{w\in\mathbb{D}} \alpha(w) G_{\mathbb{D}}(z,w), \qquad z\in\mathbb{D},$$

where $G_{\mathbb{D}}$ is the Green function for the unit disc,

$$G_{\mathbb{D}}(z,w) = \log\left|\frac{z-w}{1-\bar w z}\right|, \qquad z,w\in\mathbb{D}.$$

If X and Y are complex manifolds, α is a non-negative function on X, and $\Phi: Y\to X$ is a holomorphic map, then the pullback $\Phi^* u = u\circ\Phi$ satisfies $\nu_{\Phi^* u}\ge \Phi^*\nu_u$, so $\Phi^* u\in\tilde{\mathcal{F}}_{\Phi^*\alpha}$ for every $u\in\tilde{\mathcal{F}}_\alpha$. This implies $\Phi^*\tilde G_\alpha \le \tilde G_{\Phi^*\alpha}$, i.e., $\tilde G_\alpha(x)\le \tilde G_{\Phi^*\alpha}(y)$ if $x=\Phi(y)$, and in particular

$$\tilde G_\alpha(x) \le \tilde G_{f^*\alpha}(0) = \sum_{w\in\mathbb{D}} f^*\alpha(w)\log|w|, \qquad f\in\mathcal{O}(\mathbb{D},X), \quad f(0)=x.$$

One of the main results of [11] and [13] is that for every manifold X and every non-negative function α we have the formula

$$\tilde G_\alpha(x) = \inf\{\tilde G_{f^*\alpha}(0)\,;\ f\in\mathcal{O}(\overline{\mathbb{D}},X), f(0)=x\}, \qquad x\in X. \tag{1.1}$$

Here $\mathcal{O}(\mathbb{D},X)$ is the family of all analytic discs in X and $\mathcal{O}(\overline{\mathbb{D}},X)$ is the subclass of closed analytic discs, i.e., maps from $\mathbb{D}$ to X that can be extended to holomorphic maps in some

neighbourhood of the closed disc $\overline{\mathbb{D}}$. Results of this kind originate in Poletsky's theory of analytic disc functionals, started in [19] and [20].

A natural way of describing the zero set of a holomorphic function φ is to state that its germs $(\varphi)_x$ are in the stalk $\mathcal{I}_{A,x}$ of a prescribed coherent ideal sheaf $\mathcal{I}_A = (\mathcal{I}_{A,x})_{x\in X}$ of a closed complex subspace A of X. Then, if $\psi_1, \dots, \psi_m$ are local generators of $\mathcal{I}_A$ near the point a, the function φ can be represented as $\varphi = \varphi_1\psi_1 + \cdots + \varphi_m\psi_m$ near a, which implies that $\log|\varphi| \le \log|\psi| + C$ near a, where $\psi = (\psi_1, \dots, \psi_m)$, $|\cdot|$ is the euclidean norm, and C is a constant depending on φ and the generators.

We define $\mathcal{F}_A$ as the class of all negative plurisubharmonic functions u in X satisfying $u \le \log|\psi| + O(1)$ locally in X, and we define the function G_A, the *pluricomplex Green function with singularities along* A, as the supremum over the class $\mathcal{F}_A$.

It follows from the definition of G_A that if A' is the restriction of A to a domain X' in X, φ is holomorphic function on X, and $(\varphi)_x \in \mathcal{I}_{A,x}$ for all $x \in X'$, then

$$|\varphi| \le e^{G_{A'}(x)} \sup_{X'} |\varphi|, \quad x \in X',$$

which is a variant of the Schwarz lemma for the ideal sheaves.

In order to relate G_A to the Green functions $\tilde{G}_\alpha$ above, we define the function $\tilde{\nu}_A$ on X by $\tilde{\nu}_A(x) = \nu_{\log|\psi|}(x)$ if $\psi = (\psi_1, \dots, \psi_m)$ are local generators for $\mathcal{I}_A$ in some neigbourhood of x. It is easy to see that $\tilde{\nu}_A(x)$ is independent of the choice of the generators (actually, it equals the minimal multiplicity of the functions from $\mathcal{I}_{A,x}$ at x), so $\tilde{\nu}_A$ is a well defined function on X and $\nu_u \ge \tilde{\nu}_A$ for all $u \in \mathcal{F}_A$. Hence, with $\tilde{\nu}_A$ in the role of α above, we have $\mathcal{F}_A \subseteq \tilde{\mathcal{F}}_{\tilde{\nu}_A}$ which implies

$$G_A \le \tilde{G}_{\tilde{\nu}_A}.$$

In general, $G_A \ne \tilde{G}_{\tilde{\nu}_A}$ as seen from the example where $X = \mathbb{D}^2$ and $\mathcal{I}_A$ has the global generators $\psi = (\psi_1, \psi_2)$ with $\psi_1(z) = z_1^2$ and $\psi_2(z) = z_2$. Then $G_A(z) = \max\{2\log|z_1|, \log|z_2|\}$ and $\tilde{G}_{\tilde{\nu}_A}(z) = \max\{\log|z_1|, \log|z_2|\}$ for $z = (z_1, z_2) \in \mathbb{D}^2$. If, on the other hand, A is an effective divisor generated by the function ψ in an open subset U of X, then by [12], Prop. 3.2, the function $\tilde{G}_{\tilde{\nu}_A} - \log|\psi|$ on $U \setminus |A|$ can be extended to a plurisubharmonic function on U. This implies that $G_A = \tilde{G}_{\tilde{\nu}_A}$ for effective divisors A.

Now to the content of the paper. In Section 2 we present the main results, which are proved in later sections. Our first task is to prove that $G_A \in \mathcal{F}_A$. In Section 2 we show how this follows from the facts that $\tilde{G}_\alpha \in \tilde{\mathcal{F}}_\alpha$ for all $\alpha : X \to [0, +\infty)$, $G_A = \tilde{G}_{\tilde{\nu}_A}$ if A is an effective divisor, and a variant of the Hironaka desingularization theorem. By the same desingularization technique we establish a representation of the Green function as the lower envelope of the analytic disc functional $f \mapsto G_{f^*A}(0)$. In Section 3 we study decomposition in ideal sheaves as a preparation for Section 4 where we prove that the estimates in the definition of the class $\mathcal{F}_A$ are locally uniform. This gives a direct proof of the relation $G_A \in \mathcal{F}_A$ (without referring to desingularization), which in turn implies certain refined maximality properties of the Green function. In Section 5 we get a representation for the current $(dd^cG_A)^p$ in the case when the ideal sheaf $\mathcal{I}_A$ has global generators, and in Section 6 we study the case when the space is reduced. In Section 7 we prove the product property of Green functions, and finally in Section 8 we give a few explicit examples.

2 Definitions and main results

We shall always let X be a complex manifold and assume that X is connected. We denote by $\mathrm{PSH}(X)$ the class of all plurisubharmonic functions on X and by $\mathrm{PSH}^-(X)$ its subclass of all non-positive functions. We take $-\infty \in \mathrm{PSH}(X)$ and set $\nu_{-\infty} = +\infty$. We let $\mathcal{O}_X$ denote the sheaf of germs of locally defined holomorphic functions on X. We let A be a closed complex subspace of X, $\mathcal{I}_A = (\mathcal{I}_{A,x})_{x\in X}$ be the associated coherent sheaf of ideals in $\mathcal{O}_X$, and $|A|$ be the analytic variety in X defined as the common set of zeros of the locally defined functions on X with germs in $\mathcal{I}_A$. If U is an open subset of X, then we let $\mathcal{I}_{A,U}$ denote the space of all holomorphic functions on U with germs in $\mathcal{I}_A$. We let $\mathbb{D}$ denote the open unit disc in the complex plane $\mathbb{C}$ and $\mathbb{T}$ denote the unit circle. We let $\mathcal{O}(Y,X)$ denote the set of all holomorphic maps from a complex manifold Y into X. A map in $\mathcal{O}(\mathbb{D},X)$ is called an *analytic disc*, and if it can be extended to a holomorphic map in some neighbourhood of the closed disc $\overline{\mathbb{D}}$ then it is said to be *closed*. The collection of all closed analytic discs is denoted by $\mathcal{O}(\overline{\mathbb{D}},X)$.

Definition 2.1 Given a complex subspace A of a connected complex manifold X, the class $\mathcal{F}_A$ consists of all functions $u \in \mathrm{PSH}^-(X)$ such that for every point $a \in X$ there exist local generators $\psi_1,\dots,\psi_m$ for $\mathcal{I}_A$ near a and a constant C depending on u and the generators with $u \le \log|\psi| + C$ near a.

Observe that $-\infty \in \mathcal{F}_A$ for every A.

Definition 2.2 The *pluricomplex Green function G_A with singularities along* A is the upper envelope of all the functions from the class $\mathcal{F}_A$, i.e.,

$$G_A(x) = \sup\{u(x);\ u \in \mathcal{F}_A\}, \qquad x \in X.$$

The local estimate $u \le \log|\psi| + C$ is independent of the choice of generators, i.e., if we have another set of generators $\psi' = (\psi'_1,\dots,\psi'_k)$, then $u \le \log|\psi'| + C'$ for some constant C'. Furthermore, in the definition of the class $\mathcal{F}_A$, $\psi = (\psi_1,\dots,\psi_m)$ can be replaced by any holomorphic $\xi = (\xi_1,\dots,\xi_l)$, defined near a and satisfying

$$\log|\xi| + c_1 \le \log|\psi| \le \log|\xi| + c_2,$$

which means precisely that the integral closure of the ideal generated by the germs of the functions ξ_i at x coincides with the integral closure of the ideal $\mathcal{I}_{A,x}$ for all x in some neighbourhood of a. (See [6], Ch. VIII, Cor. 10.5.) We occasionally write $\log|\xi| \asymp \log|\psi|$ when inequalities of this kind hold.

Let X and Y be complex manifolds and $\Phi : Y \to X$ be a holomorphic map. If A is a complex subspace of X, then we have a natural definition of a pullback Φ^*A of A as a complex subspace of Y. The ideal sheaf $\mathcal{I}_{\Phi^*A}$ is locally generated at a point b by $\Phi^*\psi_1,\dots,\Phi^*\psi_m$ if $\psi_1,\dots,\psi_m$ are local generators for $\mathcal{I}_A$ at $\Phi(b)$. It is evident that $\Phi^*u \in \mathcal{F}_{\Phi^*A}$ for all $u \in \mathcal{F}_A$, so

$$\Phi^*G_A \le G_{\Phi^*A}. \tag{2.1}$$

If Φ is proper and surjective and $v : Y \to \mathbb{R}\cup\{-\infty\}$ is an upper semi-continuous function, then the push-forward $\Phi_* v$ of v to X is well defined by the formula

$$\Phi_* v(x) = \max_{y\in\Phi^{-1}(x)} v(y), \qquad x \in X.$$

Proposition 2.3 *Let X and Y be complex manifolds of the same dimension and $\Phi : Y \to X$ be a proper surjective holomorphic map (for example, a finite branched covering). Then $\Phi_* v \in \mathrm{PSH}(X)$ for all $v \in \mathrm{PSH}(Y)$.*

Proof: In order to show that $\Phi_* v$ is upper semicontinuous, we need to prove the relation $\Phi_* v(a) \geq \limsup_{x\to a} \Phi_* v(x)$ for every $a \in X$. We take a sequence $a_j \to a$ such that $\Phi_* v(a_j) \to \limsup_{x\to a} \Phi_* v(x)$. Since v is upper semicontinuous and Φ is proper, there exist $b_j \in \Phi^{-1}(a_j)$ such that $v(b_j) = \Phi_* v(a_j)$. By replacing (b_j) by a subsequence we may assume that $b_j \to b \in Y$. Then $\Phi(b) = a$ and

$$\Phi_* v(a) \geq v(b) \geq \limsup_{j\to+\infty} v(b_j) = \lim_{j\to+\infty} \Phi_* v(a_j) = \limsup_{x\to a} \Phi_* v(x).$$

We let V denote the set of all points y in Y for which $d_y\Phi$ is degenerate. Then V is an analytic variety in Y and Remmert's proper mapping theorem implies that $W = \Phi(V)$ is an analytic variety in X. It is sufficient to show that $\Phi_* v$ is plurisubharmonic in a neighbourhood of every point $a \in X \setminus W$, for the upper semicontinuity of $\Phi_* v$ then implies that $\Phi_* v \in \mathrm{PSH}(X)$.

Since Φ is a local biholomorphism on $Y \setminus \Phi^{-1}(W)$, it follows that the fiber $\Phi^{-1}(a)$ is discrete and compact, thus finite, say that it consists of the points $b_1, \dots, b_m$. We choose a neighbourhood U of a in $X \setminus W$ and biholomorphic maps $F_j : U \to F_j(U) \subseteq Y \setminus \Phi^{-1}(W)$ with $F_j(a) = b_j$. Then $\Phi_* v(x) = \sup_{1\leq j\leq m} v \circ F_j(x)$ for all $x \in U$, which shows that $\Phi_* v$ is plurisubharmonic in U. ■

It is obvious that $u = \Phi_*\Phi^* u$ for all $u \in \mathrm{PSH}(X)$ and $v \leq \Phi^*\Phi_* v$ for all $v \in \mathrm{PSH}(Y)$.

Proposition 2.4 *Let X and Y be complex manifolds of the same dimension, A be a closed complex subspace of X, and $\Phi : Y \to X$ be a proper surjective holomorphic map. Then $\Phi_* v \in \mathcal{F}_A$ for all $v \in \mathcal{F}_{\Phi^* A}$ and*

$$\Phi^* G_A = G_{\Phi^* A}.$$

Proof: If $a \in X$ and $\psi_1, \dots, \psi_m$ are local generators for $\mathcal{I}_A$ near a, then $v \leq \Phi^* \log|\psi| + C$ in some neighbourhood of the compact set $\Phi^{-1}(a)$, which implies $\Phi_* v \leq \log|\psi| + C$ near a. Hence we conclude from Prop. 2.3 that $\Phi_* v \in \mathcal{F}_A$. Since $\Phi^* G_A \leq G_{\Phi^* A}$, it is sufficient to prove that $v \leq \Phi^* G_A$ for every $v \in \mathcal{F}_{\Phi^* A}$. We have $\Phi_* v \in \mathcal{F}_A$, so $v \leq \Phi^*\Phi_* v \leq \Phi^* G_A$. ■

Our first main result is

Theorem 2.5 *If X is a complex manifold and A is a closed complex subspace of X, then $G_A \in \mathcal{F}_A$.*

Observe that in our definition of the class $\mathcal{F}_A$, the constant C in the local estimates $u \le \log|\psi| + C$ is allowed to depend both on the function u and the local generators. The main work in our proof of Theorem 2.5 in Sections 3 and 4 is to prove that these estimates are indeed locally uniform, i.e., we show that if U is the domain of definition of ψ and K is a compact subset of U, then there exists a constant C_K, only depending on K and ψ, such that $u \le \log|\psi| + C_K$ on K for all $u \in \mathcal{F}_A$. (See Lemma 4.2.)

Let us now show how Theorem 2.5 follows from the facts that $\tilde{G}_\alpha \in \tilde{\mathcal{F}}_\alpha$ for all $\alpha : X \to [0, +\infty)$, $G_A = \tilde{G}_{\tilde{\nu}_A}$ if A is an effective divisor, and the following variant of the Hironaka desingularization theorem. (See [1], Theorems 1.10 and 13.4.)

Given a closed complex subspace A on a manifold X, there exists a complex manifold $\hat{X}$ and a proper surjective holomorphic map $\Phi : \hat{X} \to X$ which is an isomorphism outside $\Phi^{-1}(|A|)$ and such that $\hat{A} = \Phi^* A$ is a normal-crossing principal ideal sheaf (i.e., generated locally by a monomial in suitable coordinates).

If we let Φ denote the desingularization map, then

$$G_A = \Phi_*\Phi^* G_A = \Phi_* G_{\Phi^* A} = \Phi_* G_{\hat{A}} = \Phi_* \tilde{G}_{\nu_{\hat{A}}}.$$

Since $\tilde{G}_{\tilde{\nu}_{\hat{A}}} \in \mathcal{F}_{\hat{A}}$, Proposition 2.4 gives $G_A \in \mathcal{F}_A$ and the theorem is proved.

If X is one-dimensional, i.e., a Riemann surface, then $\mathcal{I}_A$ is a principal ideal sheaf. If $\mathcal{I}_A = 0$, the zero sheaf, then $\tilde{\nu}_A = +\infty$. If $\mathcal{I}_A \ne 0$, then $|A|$ is discrete, for $a \notin |A|$ we have $\mathcal{I}_{A,a} = \mathcal{O}_{X,a}$ and $\tilde{\nu}_A(a) = 0$, and for $a \in |A|$ the ideal $\mathcal{I}_{A,a}$ is generated by the germ of ζ_a^m at a, where $m = \tilde{\nu}_A(a) > 0$ and ζ_a is a local generator for $\mathcal{I}_A$ near a with $\zeta_a(a) = 0$. We obviously have

$$G_A \ge \sum_{a \in \mathbb{D}} \tilde{\nu}_A(a) G_X(\cdot, a) \in \mathcal{F}_A,$$

where $G_X(\cdot, a)$ is the Green function on X with single pole at a. In the special case $X = \mathbb{D}$, every function $u \in \mathcal{F}_A \setminus \{-\infty\}$ can be represented by the Poisson–Jensen formula

$$u(z) = \frac{1}{2\pi} \int_{\mathbb{D}} G_{\mathbb{D}}(z, \cdot)\, \Delta u + \int_{\mathbb{T}} P_{\mathbb{D}}(z, t)\, d\lambda_u(t), \qquad z \in \mathbb{D},$$

where $P_{\mathbb{D}}$ is the Poisson kernel for the unit disc $\mathbb{D}$ and λ_u is a nonpositive measure on the unit circle $\mathbb{T}$ (the boundary value of u). We have $\nu_u(a) = \Delta u(\{a\})/2\pi$, so $\Delta u \ge 2\pi \sum_{a \in \mathbb{D}} \tilde{\nu}_A(a)\delta_a$, where δ_a is the Dirac measure at the point a. Thus the Poisson–Jensen formula implies

$$u(z) \le \int_{\mathbb{D}} G_{\mathbb{D}}(z, \cdot) \left(\sum_{a \in \mathbb{D}} \tilde{\nu}_A(a)\delta_a \right) = \sum_{a \in \mathbb{D}} \tilde{\nu}_A(a) G_{\mathbb{D}}(z, a)$$

and we conclude that

$$G_A(z) = \sum_{a \in \mathbb{D}} \tilde{\nu}_A(a) G_{\mathbb{D}}(z, a) = \sum_{a \in \mathbb{D}} \tilde{\nu}_A(a) \log\left| \frac{z - a}{1 - \bar{a}z} \right|, \qquad z \in \mathbb{D},$$

for every closed complex subspace A of $\mathbb{D}$.

Now we let X be any manifold, $f \in \mathcal{O}(\mathbb{D}, X)$ be an analytic disc, and $a \in \mathbb{D}$. If $\psi_1, \dots, \psi_m$ are local generators for A at $f(a)$, then $f^*\psi_1, \dots, f^*\psi_m$ are local generators for $\mathcal{I}_{f^*A}$ near a.

If all these functions are zero in some neighbourhood of A, then $\mathcal{I}_{f^*A} = 0$ and $\tilde{\nu}_{f^*A} = +\infty$. If one of them is not zero at a, then $\mathcal{I}_{f^*A,a} = \mathcal{O}_{X,a}$ and $\tilde{\nu}_{f^*A}(a) = 0$, and if they have a common isolated zero at a, then $\tilde{\nu}_{f^*A}(a)$ is the smallest positive multiplicity of them. Since $f^*G_A \leq G_{f^*A}$, we get

$$G_A(x) \leq G_{f^*A}(0) = \sum_{a\in\mathbb{D}} \tilde{\nu}_{f^*A}(a) \log|a|, \qquad f \in \mathcal{O}(\mathbb{D}, X), \quad f(0) = x.$$

Theorem 2.6 *Let X be a complex manifold and A be a closed complex subspace of X. Then*

$$G_A(x) = \inf\{G_{f^*A}(0)\,;\, f \in \mathcal{O}(\overline{\mathbb{D}}, X), x = f(0)\}, \qquad x \in X.$$

Let us show how the theorem follows from Hironaka's desingularization theorem. If we use the disc formula (1.1) for $\tilde{G}_\alpha$ with $\alpha = \tilde{\nu}_{\hat{A}}$, the fact that $G_{\hat{A}} = \tilde{G}_\alpha$, and the desingularization map Φ above with $x = \Phi(\hat{x})$, then

$$\begin{aligned} G_A(x) = \Phi^*G_A(\hat{x}) = G_{\hat{A}}(\hat{x}) &= \inf\{\tilde{G}_{g^*\alpha}(0)\,;\, g \in \mathcal{O}(\overline{\mathbb{D}}, \hat{X}), g(0) = \hat{x}\} \\ &\geq \inf\{G_{g^*\hat{A}}(0)\,;\, g \in \mathcal{O}(\overline{\mathbb{D}}, \hat{X}), g(0) = \hat{x}\} \\ &= \inf\{G_{g^*\Phi^*A}(0)\,;\, g \in \mathcal{O}(\overline{\mathbb{D}}, \hat{X}), g(0) = \hat{x}\} \\ &= \inf\{G_{(\Phi_* g)^*A}(0)\,;\, g \in \mathcal{O}(\overline{\mathbb{D}}, \hat{X}), g(0) = \hat{x}\} \\ &\geq \inf\{G_{f^*A}(0)\,;\, f \in \mathcal{O}(\overline{\mathbb{D}}, X), f(0) = x\} \end{aligned}$$

and we have proved Theorem 2.6. We will prove this theorem without reference to desingularization or the disc formula for $\tilde{G}_\alpha$ in a separate paper.

Let X_1 and X_2 be complex manifolds, A_1 and A_2 be closed complex subspaces of X_1 and X_2, respectively, $X = X_1 \times X_2$ be the product manifold of X_1 and X_2, and $A = A_1 \times A_2$ be the product space of A_1 and A_2. If $a = (a_1, a_2) \in X$ and $\psi_1^1, \dots, \psi_k^1$ and $\psi_1^2, \dots, \psi_l^2$ are local generators for $\mathcal{I}_{A_1}$ and $\mathcal{I}_{A_2}$ near a_1 and a_2, respectively, then the functions

$$x = (x_1, x_2) \mapsto \psi_1^1(x_1), \dots, \psi_k^1(x_1), \psi_1^2(x_2), \dots, \psi_l^2(x_2).$$

are generators for $\mathcal{I}_A$ near a. This implies that $X \ni x = (x_1, x_2) \mapsto \max\{u_1(x_1), u_2(x_2)\}$ is in $\mathcal{F}_A$ for all $u_1 \in \mathcal{F}_{A_1}$ and $u_2 \in \mathcal{F}_{A_2}$, so we obviously have

$$G_A(x) \geq \max\{G_{A_1}(x_1), G_{A_2}(x_2)\}, \qquad x = (x_1, x_2) \in X.$$

The following is called the *product property* for Green functions.

Theorem 2.7 *Let X_1 and X_2 be complex manifolds, A_1 and A_2 be closed complex subspaces of X_1 and X_2, respectively, and A be the product of A_1 and A_2 in $X = X_1 \times X_2$. Then*

$$G_A(x) = \max\{G_{A_1}(x_1), G_{A_2}(x_2)\}, \qquad x = (x_1, x_2) \in X.$$

We base our proof on Th. 2.6 and give it in Section 7.

It was shown in [12], Prop. 3.2, that if A is given by a single holomorphic function with effective divisor Z_A, then G_A satisfies $dd^cG_A \geq Z_A$ and, moreover, it is the largest negative

plurisubharmonic function with this property. (See the last statement of Th. 3.3 in [12]). Here $d = \partial + \bar\partial$, $d^c = (\partial - \bar\partial)/2\pi i$.

In the general case, a space A generates holomorphic chains

$$Z_A^p = \sum_i m_{i,p}[A_i^p], \tag{2.2}$$

where A_i^p are p-codimensional components of $|A|$ and $m_{i,p} \in \mathbb{Z}^+$. Namely, if on a domain $U \subset X$ the space A is given by functions $\psi_1, \dots, \psi_m$ and $\operatorname{codim}|A| = p$ there, then by the King-Demailly formula ([5], Th. 6.20),

$$(dd^c \log|\psi|)^p = \sum_i m_{i,p}[A_i^p] + R \quad \text{on } U,$$

where $m_{i,p}$ is the generic multiplicity of ψ along A_i^p and R is a positive closed current of bidegree (p,p) on U, such that $\chi_{|A|}R = 0$ and $\operatorname{codim} E_c(R) > p$ for every $c > 0$. Here χ_S is the characteristic function of a set S, $E_c(R) = \{x\,;\, \nu_R(x) \ge c\}$ and $\nu_R(x)$ is the Lelong number of the current R at x. In other words, the holomorphic chain Z_A^p given by (2.2) is the residual Monge-Ampère current of $\log|\psi|$ on $|A| \cap U$.

Theorem 2.8 *Let A have bounded global generators ψ in X. Then*

(i) $G_A = \log|\psi| + O(1)$ locally near $|A|$.

(ii) If $\operatorname{codim}|A| = p$ on $U \subset X$, then $(dd^c G_A)^p = Z_A^p + Q$ on U, where Q is a positive closed current of bidegree (p,p) on U, such that $\chi_{|A|}Q = 0$ and $\operatorname{codim} E_c(Q) > p$ for every $c > 0$. If $U \cap |A| \subset J^p$, then Q has zero Lelong numbers; here the set J^p consists of all points $a \in |A|$ such that p is the minimal number of generators of a subideal of $\mathcal{I}_{A,a}$ whose integral closure is equal to the integral closure of $\mathcal{I}_{A,a}$.

A proof is given in Section 5 (and the sets J^p are introduced and studied in Section 3).

3 Decomposition in ideal sheaves

In the case of complete intersection, i.e., when for every $a \in |A|$ the local ideal $\mathcal{I}_{A,a}$ is generated by precisely $p = \operatorname{codim}_a|A|$ germs of holomorphic functions, the relation $G_A \in \mathcal{F}_A$ is in fact quite easy to prove without using the desingularization technique. The main result of this section, Prop. 3.5, gives a tool for the reduction of the general situation to the complete intersection case in Section 4. Our approach develops a method from [21].

We recall some basics on complex Grassmannians. (See, e.g., [2], A3.4-5.) The Grassmannian $G(k,m)$ is the set of all k-dimensional linear subspaces of $\mathbb{C}^m$ with the following complex structure. Let $S_{1\dots k}$ be the set of all $L \in G(k,m)$ whose projections to the coordinate plane $\mathbb{C}_{1\dots k}$ of the variables $z_1, \dots, z_k$ are bijective. Choosing a basis $\{(e_j, w_j)\}$ in $L \in S_{1\dots k}$ with e_j the standard basis vectors in $\mathbb{C}^k$ and w_j vectors in $\mathbb{C}^{m-k}$, we get a representation of L as the $k \times m$-matrix (E, W), where E is the unit $k \times k$-matrix and W is a $k \times (m-k)$-matrix. This gives a parametrization of $S_{1\dots k}$ by $k \times (m-k)$-matrices W. In a similar way we parametrize all the charts S_I, $I = (i_1, \dots, i_k)$. Since the neighbouring relations are holomorphic, this

determines a complex structure on $G(k,m)$. It is easy to see that $\dim G(k,m) = k(m-k)$. The set $\{(z,L)\,;\, z\in L\}\subset \mathbb{C}^m\times G(k,m)$ is sometimes called the *incidence manifold*.

Let $\psi:\Omega\to\mathbb{C}^m$, $m>1$, be a holomorphic map on a domain Ω in $\mathbb{C}^n$ and $Z=\{x\in\Omega\,;\,\psi(x)=0\}$. If U is a subdomain of Ω, then the graph $\Gamma_U=\{(x,\psi(x))\,;\,x\in U\}$ of ψ over U is an n-dimensional complex manifold in $\mathbb{C}^n\times\mathbb{C}^m$. Given $k\le m-1$, let Γ_U^k be the pullback of Γ_U to the incidence variety in $\Gamma_U\times G(k,m)$. Namely, Γ_U^k is the closure of the set

$$\{(x,\psi(x),L)\,;\, x\in U\setminus Z,\ L\in G(k,m),\ \psi(x)\in L\}.$$

By ρ_k we denote the projection from Γ_Ω^k to $G(k,m)$, and by π_k its projection to Ω.

For $x\in\Omega\setminus Z$, the fiber $\rho_k\circ\pi_k^{-1}(x)$ consists of all $L\in G(k,m)$ passing through $\psi(x)\neq 0$ and thus is isomorphic to $G(k-1,m-1)$. Therefore $\dim\Gamma_\Omega^k = n+(k-1)(m-k)$.

Let $I^k=I^k(\psi)$ be the collection of all points x in Ω such that $\rho_k(\Gamma_U^k)=G(k,m)$ for every neighbourhood U of x, i.e., $\rho_k\circ\pi_k^{-1}(x)=G(k,m)$. Evidently, $I^1\subseteq I^2\subseteq\ldots\subseteq I^{m-1}\subseteq Z$.

Lemma 3.1 *I^k is an analytic set of dimension at most $n-m+k-1$.*

Proof: We have

$$\pi_k^{-1}(I^k)=I^k\times\{0\}\times G(k,m), \tag{3.1}$$

so $I^k\subset\pi_k\circ\rho_k^{-1}(L)$ for each $L\in G(k,m)$. On the other hand, for every $x\notin I^k$ there exists $L\in G(k,m)$ such that $x\notin\pi_k\circ\rho_k^{-1}(L)$. Thus

$$I^k=\bigcap_{L\in G(k,m)}\pi_k\circ\rho_k^{-1}(L).$$

Each $\rho_k^{-1}(L)$ is an analytic set in Γ_Ω^k. Since the map π_k is proper, Remmert's theorem implies that $\pi_k\circ\rho_k^{-1}(L)$ is an analytic subset of Ω for any L, and so is I^k.

The set $\pi_k^{-1}(I^k)$ is a nowhere dense analytic subset of Γ_Ω^k, and thus $\dim\pi_k^{-1}(I^k) < \dim\Gamma_\Omega^k = n+(k-1)(m-k)$. By (3.1), $\dim\pi_k^{-1}(I^k)=\dim I^k+k(m-k)$. Therefore $\dim I^k<n+(k-1)(m-k)-k(m-k)=n-m+k$. ■

Corollary 3.2 *If $m>n$, then $I^k=\emptyset$ for all $k\le m-n$.*

Lemma 3.3 *For any $a\in Z\backslash I^k$ there exist a neighbourhood U of a and holomorphic functions $\xi_1,\ldots,\xi_{m-k}$ (linear combinations of $\psi_1,\ldots,\psi_m$) such that $\log|\psi|\asymp\log|\xi|$ in U.*

Proof: Given $a\in Z\setminus I^k$, one can find a neighbourhood U of a such that $\rho_k(\Gamma_U^k)\neq G(k,m)$. Since the set $G(k,m)\setminus\rho_k(\Gamma_U^k)$ is open, there exists L_0 in the chart $S_{1\ldots k}$ of $G(k,m)$ such that

$$\psi(x)\cap\omega=\emptyset \tag{3.2}$$

for some neighbourhood $\omega\subset S_{1\ldots k}$ of L_0 and all $x\in U\setminus Z$.

Let (E,W_0) be the canonical representation of L_0. For every $y=(y',y'')\in\mathbb{C}^k\times\mathbb{C}^{m-k}$ with $y'\neq 0$, the map $y\mapsto(y',y'W_0)$ is the projection to the space L_0. By elementary linear algebra arguments (see Lemma 3.4 below), relation (3.2) implies existence of $r>0$ such that

$$|\psi''(x)-\psi'(x)W_0|\ge r|\psi'(x)|,\quad x\in U. \tag{3.3}$$

We define a map $\xi : U \to \mathbb{C}^{m-k}$ by $\xi(x) = \psi''(x) - \psi'(x)W_0$. Then

$$|\xi(x)| \le C|\psi(x)|, \quad x \in U.$$

Furthermore, inequality (3.3) implies

$$|\psi|^2 \le |\psi'|^2 + 2|\psi'' - \psi' W_0|^2 + 2|\psi' W_0|^2 \le C|\psi'' - \psi' W_0|^2 = C|\xi|^2,$$

and the assertion follows. ■

Lemma 3.4 *Let W_0 be a complex $k \times (m-k)$-matrix and a set $S \subset \mathbb{C}^k \times \mathbb{C}^{m-k}$ be such that $|y'' - y'W| > 0$ for all $y = (y', y'') \in S$ and all matrices $W \in \mathbb{C}^{k(m-k)}$ with $|W - W_0| < \delta$ (all the norms $|\cdot|$ are the Euclidean norms in the corresponding linear spaces). Then*

$$|y'' - y'W| \ge \frac{\delta}{k}|y'|, \quad y \in S, \ |W - W_0| < \delta.$$

Proof: Suppose there exists $y \in S$ and W in the δ-neighbourhood of W_0 such that $|y'' - y'W| < \frac{\delta}{k}|y'|$. For the vector $z = (z', z'') := (y', y'' - y'W_0)$ this means $|z''| < \frac{\delta}{k}|z'|$.

We choose $l \in [1, k]$ such that $|z_l| = \max\{|z_i|\,;\, 1 \le i \le k\}$ and consider the $k \times (m-k)$-matrix V with the entries $V_{lj} = z_{k+j}/z_l$ for $1 \le j \le m-k$, and $V_{ij} = 0$ for all $i \ne l$ and $1 \le j \le m-k$. Then

$$|V| = \frac{|z''|}{|z_l|} \le \frac{|z''|}{k|z'|} < \delta$$

and $z'V = z''$. The latter relation is equivalent to $y'' - y'W = 0$ with $W = W_0 + V$. Since $|W - W_0| = |V| < \delta$, this contradicts the hypothesis of the lemma. ■

We recall that the *analytic spread* of an ideal $\mathcal{I}$ equals the minimal number of generators of a subideal of $\mathcal{I}$ whose integral closure coincides with the integral closure of $\mathcal{I}$, see [18].

Proposition 3.5 *Let A be a closed complex subspace of a manifold X, $\dim X = n$. Then the set $|A|$ can be decomposed into the disjoint union of local analytic varieties J^k, $1 \le k \le n$, such that*

(i) $\operatorname{codim} J^k \ge k$ *and*

(ii) for each $a \in J^k$, the ideal $\mathcal{I}_{A,a}$ has analytic spread at most k.

Proof: Let $\psi = (\psi_1, \ldots, \psi_m)$ be generators of $\mathcal{I}_A$ on a domain $\Omega \subset X$. Set $N = \min\{n, m\}$, $Z = |A| \cap \Omega$, $J^1 = Z \setminus I^{m-1}$, $J^k = I^{m-k+1} \setminus I^{m-k}$ for $k = 2, \ldots, N-1$, and $J^N = I^{m-N+1}$ (some of them can be empty). The sets J^k are pairwise disjoint, $\dim J^k \le n-k$ (Lemma 3.1), and $\cup_k J^k = Z$. On a neighbourhood of each point of J^k, the singularity of the function $\log|\psi|$ is equivalent to one defined by the function $\log|\xi|$ with $\xi = (\xi_1, \ldots, \xi_k)$ (this follows from Lemma 3.3, if $m \le n$, and Corollary 3.2, in the case $m > n$). This means that the ideal generated by the germs of ψ_i at $a \in J^k$ has analytic spread at most k.

Let $\psi' = (\psi'_1, \ldots, \psi'_{m'})$ be other generators of $\mathcal{I}_A$ on Ω; by adding some identically zero components to either ψ or ψ' we can assume $m' = m$. For any point $a \in Z \setminus I^k(\psi)$, relation

(3.2) implies existence of a neighbourhood U' of a and a plane $L_0' \in G(k,m)$ such that $\psi'(x) \cap \omega' = \emptyset$ for some neighbourhood ω' of L_0' and all $x \in U' \setminus Z$, so $a \in Z \setminus I^k(\psi')$. This shows that the sets J^k are independent of the choice of generators of $\mathcal{I}_{A,\Omega}$. Therefore each J^k is well defined as a local (not necessarily closed) analytic variety in X with properties (i) and (ii). ■

Example 3.6 Let A be generated by $\psi(x) = (x_1^2x_2, x_1^2x_3, x_1x_2x_3)$ in $\mathbb{C}^3$. Then $|A| = \mathbb{C}_{23} \cup \mathbb{C}_1$; here $\mathbb{C}_{23}$ is the coordinate plane of the variables x_2 and x_3, i.e., $\mathbb{C}_{23} = \{x_1 = 0\}$, and $\mathbb{C}_1 = \{x_2 = x_3 = 0\}$. The variety $|A|$ has the decomposition $|A| = J^1 \cup J^2 \cup J^3$ with $J^1 = \mathbb{C}_{23} \setminus (\mathbb{C}_2 \cup \mathbb{C}_3)$, $J^2 = \mathbb{C}_1^* \cup \mathbb{C}_2^* \cup \mathbb{C}_3^*$, and $J^3 = \{0\}$. Near points of J^1 we have $\log|\psi| \asymp \log|x_1|$. As to J^2, the relation $\log|\psi| \asymp \log|\xi|$ is satisfied with $\xi = (x_2, x_3)$ near points of $\mathbb{C}_1^*$, and we can take $\xi = (x_1^2, x_1x_3)$ near points of $\mathbb{C}_2^*$ and $\xi = (x_1^2, x_1x_2)$ near points of $\mathbb{C}_3^*$.

4 Upper bounds and maximality

We recall that a function $u \in \mathrm{PSH}(X)$ is called *maximal* in X if for every relatively compact subset U of X and for each upper semicontinuous function v on $\overline{U}$ such that $v \in \mathrm{PSH}(U)$ and $v \le u$ on ∂U, we have $v \le u$ in U. An equivalent form is that for any $v \in \mathrm{PSH}(X)$ the relation $\{v > u\} \Subset X$ implies $v \le u$ on X.

We will use the following variant of the maximum principle for unbounded plurisubharmonic functions.

Lemma 4.1 *Let $D \subset \mathbb{C}^k$ be a bounded domain and $u, v \in \mathrm{PSH}(D)$ such that*

(i) v is bounded above,

(ii) the set $S := v^{-1}(-\infty)$ is closed in D,

(iii) v is locally bounded and maximal on $D \setminus S$,

(iv) for any $\epsilon > 0$ there exists a compact $K_\epsilon \subset D$ such that $u(z) \le v(z) + \epsilon$ on $D \setminus K_\epsilon$, and

(v) $\limsup_{z \to a,\, z \notin S} (u(z) - v(z)) < \infty$ for each $a \in S$.

Then $u \le v$ in D.

Proof: By (i) we may assume that v is negative in D. Take any $\epsilon > 0$ and $\delta > 0$. Then it is sufficient to prove that $u_1 = (1+\delta)(u - \epsilon) \le v$. By (v) we conclude that each point $a \in S$ has a neighbourhood $U_a \Subset D$ where $u_1 \le v$ and by (iv) that there is a domain $D_1 \Subset D$ such that $u_1 \le v$ on $D \setminus D_1$. By (ii) $S \cap \overline{D}_1$ is compact, so we can take a finite covering of $S \cap \overline{D}_1$ by U_{a_j}, $1 \le j \le N$. Then $D_2 = D_1 \setminus \bigcup_j \overline{U}_{a_j}$ is an open subset of D on which v is bounded and $u_1 \le v$ holds on ∂D_2. By (iii) v is maximal on D_2, so $u_1 \le v$ on D_2 and thus on D. ■

The next statement is the crucial point in the proof that $G_A \in \mathcal{F}_A$.

Lemma 4.2 *Let $\psi = (\psi_1, \ldots, \psi_m)$ be a holomorphic map on a domain $\Omega \subset \mathbb{C}^n$ and Z be its zero set. Then for every $K \Subset \Omega$ there exists a number C_K such that any function $u \in PSH^-(\Omega)$ which satisfies $u \le \log|\psi| + O(1)$ locally near points of Z has the bound $u(x) \le \log|\psi(x)| + C_K$ for all $x \in K$.*

Proof: What we need to prove is that each point $a \in Z$ has a neighbourhood U where $u \le \log|\psi| + C$ with C independent of the function u.

Let $\operatorname{codim}_a Z = p$. Then, by Prop. 3.5, $a \in J^k$ for some $k \in [p, n]$ and thus there exist k holomorphic functions $\xi_1, \ldots, \xi_k$ such that $\log|\xi| \asymp \log|\psi|$ near a. We will argue by induction in k from p to n.

Let $a \in J^p$; this means that there is a neighbourhood V of a such that $Z \cap V$ is a complete intersection given by the functions $\xi_1, \ldots, \xi_p$. By Thie's theorem [23], (see also [5], Th. 5.8), there exist local coordinates $x = (x', x'')$, $x' = (x_1, \ldots, x_p)$, $x'' = (x_{p+1}, \ldots, x_n)$, centered at a and balls $\mathbb{B}' \subset \mathbb{C}^p$, $\mathbb{B}'' \subset \mathbb{C}^{n-p}$ such that $\mathbb{B}' \times \mathbb{B}'' \Subset V$, $Z \cap (\mathbb{B}' \times \mathbb{B}'')$ is contained in the cone $\{|x'| \le \gamma|x''|\}$ with some constant $\gamma > 0$, and the projection of $Z \cap (\mathbb{B}' \times \mathbb{B}'')$ onto $\mathbb{B}''$ is a ramified covering with a finite number of sheets. Let $r_1 = 2\gamma r_2$ with a sufficiently small $r_2 > 0$ so that $\mathbb{B}'_{r_1} \subset \mathbb{B}'$ and $\mathbb{B}''_{r_2} \subset \mathbb{B}''$, then for some $\delta > 0$

$$|\xi(x)| \ge \delta, \qquad x \in \partial\mathbb{B}'_{r_1} \times \mathbb{B}''_{r_2}.$$

Given $x''_0 \subset \mathbb{B}''_{r_2}$, denote by $Z(x''_0)$ and $\operatorname{Sing} Z(x''_0)$ the intersections of the set $\mathbb{B}'_{r_1} \times \{x''_0\}$ with the varieties Z and $\operatorname{Sing} Z$, respectively. Since the projection is a ramified covering, $Z(x''_0)$ is finite for any $x''_0 \in \mathbb{B}''_{r_2}$, while $\operatorname{Sing} Z(x''_0)$ is empty for almost all $x''_0 \in \mathbb{B}''_{r_2}$ because $\dim \operatorname{Sing} Z \le n - p - 1$; we denote the set of all such generic x''_0 by E.

Fix any $x''_0 \in E$ and consider the function

$$v(x') = \log(|\xi(x', x''_0)|/\delta).$$

It is plurisubharmonic on $\mathbb{B}'_{r_1}$, nonnegative on $\partial\mathbb{B}'_{r_1}$ and maximal on $\mathbb{B}'_{r_1} \setminus Z(x''_0)$, since the map $\xi(\cdot, x''_0) : \mathbb{B}'_{r_1} \to \mathbb{C}^p$ has no zeros outside $Z(x''_0)$.

For any function $u \in PSH^-(Y)$ which satisfies $u \le \log|\xi| + O(1)$ locally near regular points of Z, we have, by Lemma 4.1, $u(x', x''_0) < v(x')$ on the whole ball $\mathbb{B}'_{r_1}$.

Since $x''_0 \in E$ is arbitrary, this gives us $u \le \log|\xi| - \log\delta$ on $\mathbb{B}'_{r_1} \times E$. The continuity of the function $\log|\xi|$ extends this relation to the whole set $U = \mathbb{B}'_{r_1} \times \mathbb{B}''_{r_2}$, which proves the claim for $k = p$.

Now we make a step from $k-1$ to k. Since $\dim J^k \le n - k$, we use Thie's theorem to get a coordinate system centered at $a \in J^k$ such that the projection of $J^k \cap (\mathbb{B}' \times \mathbb{B}'')$ to $\mathbb{B}'' \subset \mathbb{C}^{n-k}$ is a finite map and $(\partial\mathbb{B}' \times \overline{\mathbb{B}''}) \cap J^i = \emptyset$ for all $i \ge k$. Therefore, by the induction assumption and a compactness argument, $u \le \log|\xi| + C$ near $\partial\mathbb{B}' \times \overline{\mathbb{B}''}$, where the constant C is independent of u.

Now for any $x''_0 \subset \mathbb{B}''$ we consider the function $v(x') = \log|\xi(x', x''_0)| + C$. Then Lemma 4.1 gives us $u(x', x''_0) < v(x')$ on $\mathbb{B}'$ and hence $u \le \log|\xi| + C$ on $\mathbb{B}' \times \mathbb{B}''$. ∎

Remark. Note that the uniform bound $u \le \log|\psi| + C$ near points $a \in J^p$, $\operatorname{codim}_a Z = p$, was deduced from the local bounds only near regular points of Z.

Proof of Theorem 2.5: The relation $G_A \le \log|\psi| + O(1)$ follows from Lemma 4.2. This implies that its upper semicontinuous regularization G_A^* is in $\mathcal{F}_A$ and thus $G_A^* = G_A$. ∎

One of the most important properties of the "standard" pluricomplex Green function $G_{X,a}$ with logarithmic pole at $a \in X$ is that it satisfies the homogeneous Monge-Ampère equation $(dd^c G_{X,a})^n = 0$ outside the point a; in other words, $G_{X,a}$ is a maximal plurisubharmonic function on $X \setminus \{a\}$. In our situation, one can say more.

Theorem 4.3 *The function G_A is maximal on $X \setminus |A|$ and locally maximal outside a discrete subset of $|A|$ (actually, the set J^n from Prop. 3.5). If A has $k < n$ global generators on X, then G_A is maximal on the whole X.*

Proof: Take any point $a \notin J^n$. By Proposition 3.5, there exist functions $\xi_1, \ldots, \xi_k \in \mathcal{I}_{A,U}$, $k < n$, generating an ideal whose integral closure coincides with the integral closure of $\mathcal{I}_{A,U}$, and so $G_A \le \log |\xi| + C$ on U. The function $\log |\xi|$ is maximal on U, which follows from the fact that it is the limit of the decreasing sequence of maximal plurisubharmonic functions $u_j = \frac{1}{2}\log(|\xi|^2 + \frac{1}{j})$. (See [21], Example 1.) Take any domain $W \Subset U$. Given a function $v \in PSH(U)$ with $v \le G_A$ on $U \setminus W$, we have to show that $v \le G_A$ on U. Consider the function w such that $w = G_A$ on $X \setminus W$ and $w = \max\{G_A, v\}$ on W. Since $G_A \le \log |\xi| + C$ on U, we have $w \le \log |\xi| + C$ on $U \setminus W$, and the maximality of $\log |\xi|$ on U extends this inequality to the domain W. Therefore, $w \in \mathcal{F}_A$ and thus $w \le G_A$ on U.

When $a \notin |A|$, we can take $U = X \setminus |A|$ and $\xi \equiv 1$, which gives us maximality of G_A on $U = X \setminus |A|$.

Finally, if A has $k < n$ global generators on X, then the same arguments with $U = X$ show the maximality of G_A on the whole X. ■

Remark. If $J^n = \emptyset$, the Green function is locally maximal on the whole X. We don't know if this implies its maximality on X.

5 Complex spaces with bounded global generators

If A has bounded generators ψ, which we can choose such that $|\psi| < 1$, then $\log |\psi| \in \mathcal{F}_A$. This gives immediately

Proposition 5.1 *Let A be a closed complex subspace of a manifold X and assume that A has bounded global generators $\{\psi_i\}$ (for example, X is a relatively compact domain in a Stein manifold Y and A is a restriction to X of a complex space B on Y), then*

$$G_A = \log |\psi| + O(1) \tag{5.1}$$

locally near $|A|$.

To describe the boundary behaviour of G_A, we recall the notion of strong plurisubharmonic barrier. Let X be a domain in a complex manifold Y, and let $p \in \partial X$. A plurisubharmonic function v on X is called a *strong plurisubharmonic barrier at p* if $v(x) \to 0$ as $x \to p$, while $\sup_{X \setminus V} v < 0$ for every neighbourhood V of p in Y. By standard arguments (see, e.g., [12], Proposition 2.4) we get

Proposition 5.2 *Let X be a domain in a complex manifold Y, and let a closed complex subspace of X have bounded global generators. If X has a strong plurisubharmonic barrier at $p \in \partial X \setminus |A|$, then $G_A(x) \to 0$ as $x \to p$.*

A uniqueness theorem for the Green function is similar to that for the divisor case in [12], but the proof is different (since the function $u - \log |\psi|$ need not be plurisubharmonic) and follows from Lemma 4.1 and Proposition 5.1.

Theorem 5.3 *Let a complex space A have bounded global generators ψ_i on X, and let a function $u \in PSH^-(X)$ have the properties*

(i) u is locally bounded and maximal on $X \setminus |A|$.

(ii) For any $\epsilon > 0$ there exists a compact subset K of X such that $u \geq G_A - \epsilon$ on $X \setminus K$;

(iii) $u = \log|\psi| + O(1)$ locally near $|A|$.

Then $u = G_A$.

Relation (5.1) allows us to derive the properties of the Monge-Ampére current $(dd^cG_A)^p$.

Proof of Theorem 2.8: Since G_A is locally bounded on $U \setminus |A|$ and $\operatorname{codim}|A| = p$, the current $(dd^cG_A)^p$ is well defined on U. Moreover, Siu's structural formula for positive closed currents [22] (see also [5], Theorem 6.19) gives us a (unique) representation for the current $(dd^cG_A)^p$ as

$$(dd^cG_A)^p = \sum_j \lambda_j[B_j] + Q,$$

where B_j are some irreducible analytic varieties of codimension p, λ_j are the generic Lelong numbers of $(dd^cG_A)^p$ along B_j, i.e.,

$$\lambda_j = \inf\{\nu((dd^cG_A)^p, a) : a \in B_j\},$$

and Q is a positive closed current such that $\operatorname{codim}\{x : \nu(Q, x) \geq c\} > p$ for each $c > 0$.

As G_A has asymptotics (5.1) near points of the set $|A|$, Demailly's Comparison Theorem for Lelong numbers ([5], Theorem 5.9) implies

$$\nu((dd^cG_A)^p, a) = \nu((dd^c \log|\psi|)^p, a)$$

at every point $a \in |A| \cap J^p \cap U$. In particular, the generic Lelong number of $(dd^cG_A)^p$ along each variety A_i^p equals the multiplicity of this component in $|A|$. Besides, $\nu((dd^cG_A)^p, a) = 0$ for any $a \notin |A|$. This shows that $\{B_j\}_j$ are exactly the p-codimensional components of the variety $|A|$ in U and $\sum \lambda_j[B_j] = Z_A^p$ on U.

Finally, if $U \cap |A| \subset J^p$, then $U \cap |A|$ can be given locally by p holomorphic functions ξ_i with $\log|\xi| \asymp \log|\psi|$. By King's formula, $(dd^c \log|\xi|)^p = Z_A^p$, which means, in particular, that $(dd^c \log|\xi|)^p$ has zero Lelong numbers outside $\cup_i A_i^p$. Since the currents $(dd^cG_A)^p$ and $(dd^c \log|\xi|)^p$ have the same Lelong numbers, this proves the last statement. ■

So the Green function satisfies, as in the divisor case, the relation $(dd^cG_A)^p \geq Z_A^p$, but for $p > 1$ it is not the largest negative plurisubharmonic function with this property (even for reduced spaces that are complete intersections). For example, let X be the unit polydisc in $\mathbb{C}^3$ and A be generated by $\psi(z) = (z_1, z_2)$. Then $G_A = \max\{\log|z_1|, \log|z_2|\}$ and, moreover, $(dd^cG_A)^2 = Z_A = [A]$. But the functions $u_N = \max\{N\log|z_1|, N^{-1}\log|z_2|\}$, $N > 0$, also satisfy $(dd^cu_N)^2 = [A]$, although they are not dominated by G_A. It is easy to see that the upper envelope of all such functions equals 0 outside $|A|$ and $-\infty$ on $|A|$. Therefore, in the case $\operatorname{codim}|A| > 1$ there is no counterpart for the description of the Green function in terms of the current Z_A.

6 Reduced spaces

Now we return to relations between the functions G_A and $\tilde{G}_{\tilde{\nu}_A}$ (see Introduction). As was already mentioned, one has always $G_A \le \tilde{G}_{\tilde{\nu}_A}$ and $G_A < \tilde{G}_{\tilde{\nu}_A}$ for 'generic' spaces A, however $G_A = \tilde{G}_{\tilde{\nu}_A}$ for effective divisors A. Here we show that the equality holds also in the case of *reduced* complex spaces.

When A is a reduced space, it can be identified with the analytic variety $|A|$. Its generators $\psi_1, \dots, \psi_m$ on U have the property: if a holomorphic function φ vanishes on $A \cap U$, then $\varphi = \sum h_i \psi_i$ with $h_i \in \mathcal{O}(U)$.

Since $\tilde{\nu}_A = 1$ at all regular points of A, it is natural to consider the class

$$\tilde{\mathcal{F}}_A^1 = \{u \in PSH^-(X);\ \nu_u(a) \ge 1 \text{ for all } a \in Reg\,A\}.$$

Note that upper semicontinuity of the Lelong numbers implies $\nu_u \ge 1$ on the whole A.

We evidently have $\tilde{\mathcal{F}}_A^1 \subseteq \tilde{\mathcal{F}}_{\tilde{\nu}_A} \subseteq \mathcal{F}_A$.

Theorem 6.1 *If A is a reduced subspace of X, then $\tilde{\mathcal{F}}_A^1 = \tilde{\mathcal{F}}_{\tilde{\nu}_A} = \mathcal{F}_A$ and consequently*

$$G_A(x) = \tilde{G}_{\tilde{\nu}_A}(x) = \sup\{u(x)\,;\ u \in \tilde{\mathcal{F}}_A^1\}.$$

Proof: It suffices to show that for any function $u \in \tilde{\mathcal{F}}_A^1$ and every point $a \in A$ there is a neighbourhood U of a and a constant C such that

$$u(x) \le \log|\psi(x)| + C, \qquad x \in U. \tag{6.1}$$

We will use induction on the dimension of X. The case $\dim X = 1$ is evident. Assume it proved for all X with $\dim X < n$ and take any $u \in \tilde{\mathcal{F}}_A^1$. When $\dim_a A = 0$, relation (6.1) follows easily from the fact that $\log|\psi(x)| = \log|\zeta(x)| + O(1)$ near $a \in A$, where ζ are local coordinates near a with $\zeta(a) = 0$. So we assume $\dim_a A > 0$. We first treat the case when a is a regular point of A, $\operatorname{codim}_a A = p < n$. Since the problem is local, we may then assume that $X \subset \mathbb{C}^n$ and contains the unit polydisc $\mathbb{D}^n$, $a = 0$, and the restriction A' of A to $\mathbb{D}^n$ is given by $\psi(x) = (x_1, \dots, x_p)$. Then the restriction of u to $\mathbb{D}^n$ is dominated by the Green function $\tilde{G}_{\tilde{\nu}_{A'}}$. By the product property for this type of Green function ([12], Theorem 2.5), $\tilde{G}_{\tilde{\nu}_{A'}}(x) = \max\{\log|x_j|,\ 1 \le j \le p\}$. This implies (6.1) for $a \in Reg\,A$.

For $a \in Sing\,A$ we will argue similarly to the proof of Lemma 4.2. There is a neighbourhood V of a such that $V \cap Sing\,A \subset J^p \cup J^{p+1} \cup \dots J^n$. The proof for $a \in J^k$, $p \le k \le n$, is then by induction in k.

For $a \in J^p \cap V$ relation (6.1) follows directly from the remark after Lemma 4.2.

Assuming (6.1) proved for $a \in J^p \cup \dots \cup J^k$, we take $a \in J^{k+1}$. We choose coordinates $x = (x', x'') \in \mathbb{C}^{k+1} \times \mathbb{C}^{n-k-1}$ such that $a = 0$, the projection of $J^{k+1} \cap \mathbb{B}$ to $\mathbb{B}''$ is a finite map and $\partial\mathbb{B}' \times \mathbb{B}'' \cap J^i = \emptyset$ for all $i \ge k+1$, so the k-induction assumption gives

$$u(x) \le \log|\psi(x)| + C, \quad x \in \partial\mathbb{B}' \times \mathbb{B}''. \tag{6.2}$$

Take any $b = (b', b'') \in \mathbb{B}' \times \mathbb{B}''$ and consider the $(k+1)$-dimensional plane $L = \{x\,;\ x'' = b''\}$. Then the restriction u_L of u to the plane L (in the same way we will use the denotation ψ_L, $\mathbb{B}_L$, A_L, etc.) has Lelong numbers at least 1 at all points of A_L, so $u_L \in \tilde{\mathcal{F}}^1_{A_L,\mathbb{B}_L}$. Since

$\dim \mathbb{B}_L < n$ and the components of ψ_L generate A_L, the n-induction assumption implies $u_L \in \mathcal{F}_{A_L,\mathbb{B}_L}$. Therefore, $u_L \le \log|\psi_L| + O(1)$ locally near points of A_L.

Since $a \in J^{k+1}$, we can find functions $\xi_1, \ldots, \xi_{k+1}$ such that $\log|\xi| \asymp \log|\psi|$ on $\mathbb{B}$. Therefore $u_L \le \log|\xi_L| + O(1)$ locally near all points of A_L, and, by (6.2), $u_L \le \log|\psi_L| + C_1$ on a neighbourhood of $\partial\mathbb{B}_L$ with C_1 independent of L. The function ξ_L is maximal on $\mathbb{B}_L \setminus A_L$, so by Lemma 4.1, $u_L \le \log|\xi_L| + C_1$ everywhere on $\mathbb{B}_L$. Since the plane L was chosen arbitrary, this gives us (6.1) for $a \in J^{k+1}$.

This proves the inductive step in the induction in k and, at the same time, in the induction in n. ■

Theorem 4.3 for reduced spaces has the following form (compare with the remark after the proof of Theorem 4.3).

Theorem 6.2 *The Green function of a reduced space A is maximal on $X \setminus A_0$, where A_0 is the collection of 0-dimensional components of A.*

Proof: We need to show that for every domain $U \Subset X' := X \setminus A_0$ and a function $u \in PSH(X')$ the condition $u \le G_A$ on $X' \setminus U$ implies $u \le G_A$ on U.

Consider the set $E_1(u) = \{x \in X\,;\, \nu_u(x) \ge 1\}$. Since $u \le G_A$ on $X' \setminus U$, we have $E_1(u) \setminus U \supset A \setminus U$. By Siu's theorem, E_u is an analytic variety in X, so it must contain the whole A. This means that $u \in \tilde{\mathcal{F}}^1_A$ and thus is dominated by $\tilde{G}_{\tilde{\nu}_A} = G_A$ on X. ■

7 The product property

Our proof of Th. 2.7 in this section is based on Th. 2.6. It is a modification of the proof of Th. 2.5 in [12] which in turn generalizes a proof of Edigarian [8] of the product property for the single pole Green function. For the sake of completeness we have repeated some arguments from [8], [12], and [14].

We introduce the following notation: If the function φ is holomorphic in some neighbourhood of the point a in $\mathbb{C}$, then we set $m_a(\varphi) = 0$ if $\varphi(a) \ne 0$, $m_a(\varphi) = +\infty$ if $\varphi = 0$ in some neighbourhood of a, and let $m_a(\varphi)$ be the multiplicity of a if it is an isolated zero of φ.

Lemma 7.1 *Let $x \in X$, $\alpha \in (-\infty, 0)$ and assume that $g \in \mathcal{O}(\overline{\mathbb{D}}, X)$, $g(0) = x$, and $G_{g^*A}(0) < \alpha$. Then there exist $f \in \mathcal{O}(\overline{\mathbb{D}}, X)$ and finitely many different points $a_1, \ldots, a_k \in \mathbb{D} \setminus \{0\}$ such that $f(0) = x$ and*

$$-\infty < \sum_{j=1}^{k} \tilde{\nu}_{f^*A}(a_j) \log|a_j| < \alpha. \tag{7.1}$$

Proof: We have $G_{g^*A}(0) = \sum_{a\in\mathbb{D}} \tilde{\nu}_{g^*A}(a) \log|a| < \alpha$, so we can choose finitely many points $a_1, \ldots, a_k \in \mathbb{D} \setminus \{0\}$ such that

$$\sum_{j=1}^{k} \tilde{\nu}_{g^*A}(a_j) \log|a_j| < \alpha. \tag{7.2}$$

If the sum in (7.2) is finite, we take $f = g$. If the sum is equal to $-\infty$ and $g(\mathbb{D})$ is not contained in $|A|$, then $a_j = 0$ and $0 < \tilde{\nu}_{g^*A}(a_j) < +\infty$ for some j. We choose $a \in \mathbb{D} \setminus \{0\}$ so

close to 0 that $\log|a| < \alpha$ and g is holomorphic in a neighbourhood of the image of $h : \overline{\mathbb{D}} \to \mathbb{C}$, $h(z) = z(z-a)$. If $\psi_1, \ldots, \psi_m$ are local generators for $\mathcal{I}_A$ near x, then $m_a(\psi_j \circ g \circ h) = m_0(\psi_j \circ g)$ for all j, which implies $\tilde{\nu}_{g^*A}(a) = \tilde{\nu}_{(g\circ h)^*A}(a)$. If we set $f = g \circ h$, $k = 1$, and $a_1 = a$, then $f(0) = x$ and (7.1) holds.

If the sum in (7.2) equals $-\infty$ and $g(\mathbb{D})$ is contained in $|A|$, then we may replace g by the constant disc $z \mapsto x = g(0)$. We choose a neighbourhood U of x in X and a biholomorphic map $\Phi : U \to \mathbb{D}^n$ such that $\Phi(x) = 0$. We take $v \in \mathbb{C}^n$ with $|v| < 1$ such that the disc $\overline{\mathbb{D}} \to X$, $z \mapsto \Phi^{-1}(zv)$ is not contained in $|A|$ and choose $a \in \mathbb{D} \setminus \{0\}$ so small that $\log|a| < \alpha$ and $z(z-a)v \in \mathbb{D}^n$ for all $z \in \overline{\mathbb{D}}$. If we take $k = 1$, $a_1 = a$, and let f be the map $z \mapsto \Phi^{-1}(z(z-a)v)$, then $f(0) = f(a) = x \in |A|$, $0 < \nu_{f^*A}(a) < +\infty$, and (7.1) holds. ∎

Proof of Theorem 2.7: We need to prove that $G_A(x) \leq \max\{G_{A_1}(x_1), G_{A_2}(x_2)\}$. Take $\alpha \in (-\infty, 0)$ larger than the right hand side of this inequality. It is then sufficient to show that $G_A(x) < \alpha$.

By Theorem 2.6 and Lemma 7.1 we have $f_j \in \mathcal{O}(\overline{\mathbb{D}}, X_j)$ with $f_j(0) = x_j$ and $a_{jk} \in \mathbb{D} \setminus \{0\}$, $k = 1, \ldots, l_j$, $j = 1, 2$, such that

$$-\infty < \sum_{k=1}^{l_j} \tilde{\nu}_{f_j^*A_j}(a_{jk}) \log|a_{jk}| < \alpha, \qquad j = 1, 2. \tag{7.3}$$

We choose f_j so that l_j becomes as small as possible. Then $0 < \tilde{\nu}_{f_j^*A_j}(a_{jk}) < +\infty$ and $a_{jk} \neq 0$ for all j and k. We define the Blaschke products B_j by

$$B_j(z) = \prod_{k=1}^{l_j} \left(\frac{a_{jk} - z}{1 - \bar{a}_{jk} z} \right)^{\mu_{jk}}, \qquad \text{where } \ \mu_{jk} = \nu_{f_j^*A_j}(a_{jk}).$$

Then (7.3) implies $|B_j(0)| < e^\alpha$. We set $b_j = B_j(0)$ and $\mu_j = \sum_{k=1}^{l_j} \mu_{jk}$ and we may assume that $|b_1| \geq |b_2|$. We have $B_j'(0) = B_j(0) \sum_{k=1}^{l_j} \mu_{jk}(|a_{jk}|^2 - 1)/a_{jk}$. If $B_1'(0) = 0$ we precompose f_1 with a map $\mathbb{D} \to \mathbb{D}$ which fixes the origin and makes a slight change of the points a_{1k} so that $B_1'(0) \neq 0$. By Schwarz Lemma this operation increases the value of $|b_1|$, so we still have $|b_1| \geq |b_2|$. By precomposing f_1 by a rotation, we may assume that $B_1(0) = b_1$ is not a critical value of B_1.

If c_j is one of the points a_{jk} having largest absolute value, then $|c_j| e^\beta \leq |b_j|$. For proving this inequality we assume the reverse inequality $|b_j| < |c_j| e^\beta$ and for simplicity enumerate the points so that $|a_{j1}| \leq |a_{j2}| \leq \cdots$. Then

$$\prod_{k=1}^{m_j} \left| \frac{a_{jk}}{c_j} \right|^{\mu_{jk}} < e^\beta$$

where $m_j < l_j$ is the smallest natural number with $|a_{jk}| = |c_j|$ for $k > m_j$. Hence (7.3) holds with f_j replaced by $z \mapsto f_j(c_j z)$, a_{jk} replaced by a_{jk}/c_j, and l_j by m_j, which contradicts the fact that l_j is minimal.

We may assume that $b_1 = b_2$. Indeed, if $|b_1| > |b_2|$, we choose $t \in (0, 1)$ with $t^{-\mu_2}|b_2| = |b_1|$. Then $|a_{2k}| < t$, for

$$|a_{2k}|^{\mu_2} \leq |c_2|^{\mu_2} \leq |b_2| e^{-\beta} < |b_2/b_1| = t^{\mu_2}.$$

Replacing f_2 by $z \mapsto f_2(tz)$ and a_{2k} by a_{2k}/t, we get $|b_1| = |b_2|$. Finally, replacing f_2 by $z \mapsto f_2(e^{i\theta}z)$, where $e^{i\theta\mu_2} = b_2/b_1$, and replacing a_{2k} by $e^{-i\theta}a_{2k}$, we get $b_1 = b_2$.

We let C denote the set of all critical values of B_1. We have $B_1(0) = B_2(0)$, so we can take $\varphi_2 : \mathbb{D} \to \mathbb{D} \setminus B_2^{-1}(C)$ as the universal covering map with $\varphi(0) = 0$. A theorem of Frostman, see [17], p. 33, states that an inner function on $\mathbb{D}$ omitting 0 as a non-tangential boundary value is a Blaschke product. It is easy to show, see [14], p. 272, that since $0 \notin B_2^{-1}(C)$, φ_2 satisfies the assumption in Frostman's theorem and is thus a Blaschke product. The restriction of B_1 to $\mathbb{D} \setminus B_1^{-1}(C)$ is a finite covering over $\mathbb{D} \setminus C$, so by lifting $B_2 \circ \varphi_2$ we conclude that there exists a function $\varphi_1 : \mathbb{D} \to B_1^{-1}(C)$ with $\varphi_1(0) = 0$ and $B_1 \circ \varphi_1 = B_2 \circ \varphi_2$ and Frostman's theorem implies again that φ_1 is a Blaschke product. Since $|B_j \circ \varphi_j| = 1$ almost everywhere on $\mathbb{T}$ and $B_j(0) = b_j$, we can choose $r \in (0, 1)$ such that

$$\log|B_j \circ \varphi_j(0)| - \frac{1}{2\pi}\int_0^{2\pi} \log|B_j \circ \varphi_j(re^{i\theta})|\,d\theta < \alpha.$$

We set $\sigma(z) = B_1 \circ \varphi_1(rz) = B_2 \circ \varphi_2(rz)$. By the Poisson–Jensen representation formula, the left hands side of this inequality equals $\sum_{i=1}^n \nu_i \log|z_i|$, where z_i are the zeros of σ in $\mathbb{D}$ with multiplicities ν_i for $i = 1, \dots, n$.

We define $g_j \in \mathcal{O}(\overline{\mathbb{D}}, X_j)$ by $g_j(z) = f_j \circ \varphi_j(rz)$ and $f \in \mathcal{O}(\overline{\mathbb{D}}, X)$ with $f(0) = (x_1, x_2)$ by $f = (g_1, g_2)$. If $\sigma(z_i) = 0$, then $\varphi_j(rz_i) = a_{jk_j}$ for some k_j, and

$$\nu_i = m_{z_i}(\sigma) = \mu_{jk_j} m_{z_i}(a_{jk_j} - \varphi_j(r\cdot)) = \tilde{\nu}_{f_j^*A_j}(a_{j,k_j}) m_{z_i}(a_{jk_j} - \varphi_j(r\cdot)) = \tilde{\nu}_{g_j^*A_j}(z_i)$$

Since the left hand side of this equation is independent of j, we get

$$\tilde{\nu}_{f^*A}(z_i) = \min_j \{\tilde{\nu}_{g_j^*A_j}(z_i)\} = \nu_i.$$

Hence

$$G_A(x) \le G_{f^*A}(0) = \sum_{a \in \mathbb{D}} \tilde{\nu}_{f^*A}(a)\log|a| \le \sum_{i=1}^n \tilde{\nu}_{f^*A}(z_i)\log|z_i| = \sum_{i=1}^n \nu_i \log|z_i| < \alpha.$$

■

8 Examples

Example 8.1 Let X be the unit polydisc $\mathbb{D}^n$ in $\mathbb{C}^n$, $1 \le p \le n$, and let A be generated by $\psi_k(z) = z_k^{\nu_k}$ for $1 \le k \le p$ and positive integers ν_k. Then the product property gives

$$G_A(z) = \max_{1 \le k \le p} \nu_k \log|z_k|.$$

Furthermore, we have

$$\left(dd^c G_A\right)^p = \nu_1 \cdots \nu_p\,[|A|].$$

Example 8.2 Let $X = \mathbb{D}^n$, $n \ge 2$, and let A be generated by $\psi_1(z) = z_1^2$, $\psi_2(z) = z_1 z_2$. Then

$$G_A(z) = v(z) := \log|z_1| + \max\{\log|z_1|, \log|z_2|\}, \qquad z = (z_1, z_2, z'') \in \mathbb{D}^n.$$

First we take any $z \in \mathbb{D}^n \setminus \{z_1 = 0\}$ with $|z_1| \ge |z_2|$ and consider the disc

$$f(\zeta) = (\zeta, \frac{z_2}{z_1}\zeta, z''), \quad \zeta \in \mathbb{D}.$$

Then for any $u \in \mathcal{F}_A$ we have $f^*u(\zeta) \le \log|f^*\psi(\zeta)| + C = 2\log|\zeta| + C$ and so, since $u \le 0$, $f^*u(\zeta) \le 2\log|\zeta| = f^*v(\zeta)$. As $f(z_1) = z$, this gives us $u(z) \le v(z)$.

For $z \in \mathbb{D}^n \setminus \{z_1 = 0\}$ with $|z_1| < |z_2|$, we take the disc

$$g(\zeta) = (\frac{z_1}{z_2}\zeta, \zeta, z''), \quad \zeta \in \mathbb{D}.$$

Then for any $u \in \mathcal{F}_A$ we have again $g^*u(\zeta) \le 2\log|\zeta| + C$ near the origin and, since $u(z) \le \log|z_1|$ (which is the Green function for the polydisc with the poles along the space $z_1 = 0$), $g^*u(\zeta) \le \log|z_1/z_2|$ near $\partial\mathbb{D}$. Therefore, $g^*u(\zeta) \le 2\log|\zeta| + \log|z_1/z_2| = g^*v(\zeta)$ everywhere in $\mathbb{D}$. Since $g(z_2) = z$, this shows $u(z) \le v(z)$ at all such z as well.

Note that f is, up to a Möbius transformation, an extremal disc for the disc functional $f \mapsto G_{f^*A}(0)$, while g is not. Note also that we have $dd^cG_A = [z_1 = 0] + Q$, where the current $Q = dd^c \max\{\log|z_1|, \log|z_2|\}$ has the property $Q^2 = [z_1 = z_2 = 0]$.

Example 8.3 Consider the variety $|A| = \{z_1 = z_2 = 0\} \cup \{z_2 = z_3 = 0\} \cup \{z_1 = z_3 = 0\}$ in the unit polydisk $\mathbb{D}^3$ of $\mathbb{C}^3$. It is easy to see that the corresponding reduced complex space A is generated by $\psi(z) = (z_1z_2, z_2z_3, z_1z_3)$ and that $|A|$ has the decomposition (in the sense of Prop. 3.5) $|A| = J^2 \cup J^3$ with $J^3 = \{0\}$. We claim that

$$G_A(z) = v(z) := \max\{\log|z_1z_2|, \log|z_2z_3|, \log|z_1z_3|\}.$$

It suffices to check the relation $u(z) \le v(z)$ for any function $u \in \mathcal{F}_A$ and each point $z \in \mathbb{D}^3$ with $|z_1| \ge |z_2| \ge |z_3|$, $z_2 \ne 0$. We take first any z with $|z_1| = |z_2| \ge |z_3|$ and consider the disc $f(\zeta) = \zeta z/|z_1|$, $\zeta \in \mathbb{D}$. Then $f^*u \in SH^-(\mathbb{D})$ and, since $u \le \log|\psi| + C_1$ near the origin, $f^*u(\zeta) \le \log|f^*\psi(\zeta)| + C_1 = 2\log|\zeta| + C_2$ when $|\zeta| < \epsilon$. Therefore, $f^*u(\zeta) \le 2\log|\zeta| = f^*v(\zeta)$ and, in particular, $u(z) = f^*u(|z_1|) \le f^*v(|z_1|) = v(z)$. The disc f is, up to a Möbius transformation, an extremal disc for the disc functional $f \mapsto G_{f^*A}(0)$ at such a point z.

Now we can take any z with $|z_1| > |z_2| \ge |z_3|$, $|z_2| \ne 0$, and consider the analytic disc $g(\zeta) = (z_1, \zeta z_2, \zeta z_3)$, $\zeta \in D_R$ with $R = |z_1|/|z_2| > 1$. We have $|g_1(\zeta)| = |g_2(\zeta)| \ge |g_3(\zeta)|$ when $|\zeta| = R$ and thus $g^*u \le g^*v$ on ∂D_R. Furthermore, $g^*u(\zeta) \le \log|g^*\psi(\zeta)| + C_3 \le \log|\zeta| + C_4$ near the origin. Since $g^*v(\zeta) = \log|\zeta z_1 z_2|$, this shows that $g^*u \le g^*v$ on D_R. Hence we get $u(z) = g^*u(1) \le f^*v(1) = v(z)$, which proves the claim.

The current $(dd^cG_A)^2$ has Lelong numbers equal 1 at each point $a \in J^2 = |A| \setminus \{0\}$. The point 0 is exceptional: the Lelong number $\nu(dd^cG_A, 0) = 2$, so $\nu((dd^cG_A)^2, 0) \ge 4$, while $\nu([|A|], 0) = 3$.

Example 8.4 Let X be the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$, $1 \le p \le n$, and let A be generated by $\psi_k(z) = z_k$ for all $1 \le k \le p$. In the notation $z = (z', z'')$ with $z' \in \mathbb{C}^p$ and $z'' \in \mathbb{C}^{n-p}$, the Green function

$$G_A(z) = \log\frac{|z'|}{\sqrt{1 - |z''|^2}},$$

because its restriction to every plane $z'' = c \in \mathbb{B}_{n-p}$ is the pluricomplex Green function for the ball of radius $\sqrt{1 - |c|^2}$ in $\mathbb{C}^p$ with simple pole at the origin.

Example 8.5 The Green function G_A for the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$, $n \geq 2$, with respect to A generated by $(\psi_1, \psi_2) = (z_1^2, z_2)$ is given by

$$G_A(z) = \frac{1}{2} \log \left(\frac{|z_1|^4}{(1-|z''|^2)^2} + \frac{2|z_2|^2}{1-|z''|^2} + \frac{|z_1|^2}{1-|z''|^2} \sqrt{\frac{|z_1|^4}{(1-|z''|^2)^2} + \frac{4|z_2|^2}{1-|z''|^2}} \right) - \frac{1}{2} \log 2,$$

for $z = (z_1, z_2, z'') \in \mathbb{B}_n$ (compare with the formula for the pluricomplex Green function with two poles in the ball [3]).

For proving this we let v denote the function defined by the right hand side. Then $v \in \mathrm{PSH}(\mathbb{B}_n) \cap C(\overline{\mathbb{B}}_n \setminus |A|)$ and satisfies $v(z) \leq \max\{\log |z_1|^2, \log |z_2|\} + C$ locally near $|A| = \{z_1 = z_2 = 0\}$.

Let us show that its boundary values on $\partial \mathbb{B}_n \setminus |A|$ are zero. Take any $z \in \partial \mathbb{B}_n \setminus |A|$, then $|z_1|^2 = a$, $|z_2|^2 = b$, $|z''|^2 = 1 - a - b$ with $a, b \geq 0$, $0 < a + b \leq 1$. We get

$$v(z) = \frac{1}{2} \log \left[\frac{a^2}{(a+b)^2} + \frac{2b}{a+b} + \frac{a}{a+b} \left(2 - \frac{a}{a+b} \right) \right] - \frac{1}{2} \log 2 = 0.$$

Finally we show that $v(z) \geq G_A(z)$ for almost all $z \in \mathbb{B}_n$ (which implies $v \equiv G_A$). Take any $z \in \mathbb{B}_n$ with $z_1 \neq 0$ and consider the analytic curve

$$f(\zeta) = (\zeta, \frac{z_2}{z_1^2} \zeta^2, z'').$$

Note that $f(z_1) = z$. We have $f^* v(\zeta) = 2 \log(|\zeta| / R(z))$, while $f^* G_A$ is a negative subharmonic function in the disc $|\zeta| < R(z)$ with the singularity $2 \log |\zeta|$. So $f^* G_A \leq f^* v$ and, in particular, $G_A(z) = f^* G_A(z_1) \leq f^* v(z_1) = v(z)$.

This shows also that f is, up to a Möbius transformation, an extremal disc for the disc functional $f \mapsto G_{f^* A}(0)$ at z with $z_1 \neq 0$. A corresponding extremal curve for $z = (0, z_2, z'')$ is $f(\zeta) = (0, \zeta, z'')$.

Acknowledgments. The authors thank Daniel Barlet and Alain Yger for valuable discussions. Part of the work was done during Alexander's visit to the University of Iceland and Ragnar's visit to Høgskolen i Stavanger, and the authors thank the both institutions for their kind hospitality.

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